# CHAPTER 7

## Classification of Singularities

BY


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# Module-1: Riemann's Theorem

## 1 Introduction

A point $z = z_0$ is called a regular point or an ordinary point of a function $f(z)$ if $f(z)$ is analytic at $z_0$, otherwise $z_0$ is called a singular point or a singularity of the function $f(z)$. Basically, there are two types of singularities : (i) isolated singularity; (ii) non-isolated singularity.

**Isolated Singularity**

A point $z = z_0$ is said to be an isolated singularity of a function $f(z)$ if there exists a deleted neighbourhood of $z_0$ in which the function is analytic. In other words, a point $z = z_0$ is said to be an isolated singularity of a function $f(z)$ if there exists a neighbourhood of $z_0$ which contains no other singular point of $f(z)$ except $z_0$.

For the function $f(z) = 1/z$, $z = 0$ is an isolated singular point, since $f(z)$ is analytic in the open disc $0 <| z |< r$, $r > 0$, and for $g(z) = \frac{1}{(z-1)(z-2)}$, $z = 1, 2$ are isolated singular points since the function is analytic in the annular region $1 <| z |< 2$.

**Non-isolated Singularity**

A point $z = z_0$ is called non-isolated singularity of a function $f(z)$ if every neighbourhood of $z_0$ contains at least one singularity of $f(z)$ other than $z_0$.

For the function $f(z) = Log\ z$, the principal logarithm, $z = 0$ is a non-isolated singularity, and moreover $(-\infty, 0]$ is the set of all non-isolated singularities of the function. Also, for $g(z) = 1/\sin(1/z)$, $z = 1/n\pi$, $n \in \mathbb{I}$ are the singular points, while 0 is non-isolated singularity as each neighbourhood of $z = 0$ contains a singularity of $g(z)$.

Isolated singularities are classified into (i) removable singularity; (ii) pole; and (iii) essential singularity. If $z_0$ is an isolated singularity of $f(z)$, then in some deleted neighbourhood of $z_0$ the function $f(z)$ is analytic and hence its Laurent series expansion exists as

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}, \quad 0 <\mid z-z_0 \mid< r,$$

where $r$ is the distance from $z_0$ to the nearest singularity of $f(z)$ other than $z_0$ itself. If $z_0$ is the only singularity, then $r = \infty$. The portion of the series involving negative powers of $z - z_0$, i.e. $\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$ is called the principal part of $f$ at $z_0$, while the series of non-negative powers of $z - z_0$, i.e. $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is called the regular part of $f$ at $z_0$.

**Removable singularity**

If all the coefficients $b_n$ in the principal part are zero, then $z_0$ is called a removable singularity of $f$. In this case we can make $f$ regular in $\mid z - z_0 \mid< r$ by suitably defining its value at $z_0$.

As for example, we consider the function

$$f(z) = \begin{cases} \frac{\sin\ z}{z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

The function is analytic everywhere except at $z = 0$. The Laurent expansion about $z = 0$ has the form

$$\begin{aligned} f(z) &= \frac{\sin\ z}{z} \\ &= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ \ldots\right] \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ \ldots \end{aligned}$$

Since no negative power of $z$ appears, the point $z = 0$ is a removable singularity of $f$.

**Pole**

If the principal part of $f$ at $z_0$ contains a finite number of term, then $f$ is said to have a pole at $z_0$. If $b_m$ ($m \geq 1$) is the last non-vanishing coefficient in the principal part then we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ \ldots + \frac{b_m}{(z-z_0)^m}, \ 0 <\mid z-z_0 \mid< r,$$

and the pole is said to be of order $m$. If $m = 1$, then we call the pole as a simple pole.

The function

$$\begin{aligned} f(z) &= \frac{z^2 - 3z + 4}{z - 3} \\ &= 3 + (z-3) + \frac{4}{z-3}, \ (z \neq 3) \end{aligned}$$

has a simple pole at $z = 3$.

Also the function

$$f(z) = \frac{e^z}{(z-2)^2}$$

has a pole of order 2 at $z = 2$, since

$$\begin{aligned} f(z) &= \frac{e^z}{(z-2)^2} = \frac{e^2 e^{z-2}}{(z-2)^2} \\ &= \frac{e^2}{(z-2)^2} + \frac{e^2}{z-2} + \frac{e^2}{2!} + \frac{e^2}{3!}(z-2) + \ \ldots, \quad 0 <\mid z-2 \mid< \infty. \end{aligned}$$

**Essential singularity**

If the principal part of $f$ at $z_0$ contains infinitely many nonzero terms, then $z_0$ is called an essential singularity of $f$.

As for example, the function

$$\begin{aligned} f(z) &= e^{1/z} \\ &= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ \ldots \ + \frac{1}{n!z^n} + \ \ldots, \ 0 <\mid z \mid< \infty, \end{aligned}$$

has an essential singularity at $z = 0$.

**Remark 1.** *Let us consider the expression*

$$\sum_{n=0}^{\infty} \frac{z^n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}, \quad 1 <\mid z \mid< 3.$$

*This expression has infinite number of negative powers of $z$. Even then, $z = 0$ is not an essential singularity. This is because the region of convergence is not a deleted neighbourhood of the origin. In fact, it is the Laurent expansion of the function $\frac{2z}{(1-z)(z-3)}$ in the annular region $1 <\mid z \mid< 3$. Actually, $f$ has simple poles at $z = 1$ and $z = 3$.*

**Alternate Definition of Removable singularity, Pole and Essential singularity**

A singular point $z_0$ of the function $f(z)$ is called a removable singularity of $f(z)$ if $\lim_{z \to z_0} f(z)$ exists finitely.

A singular point $z_0$ of the function $f(z)$ is called a pole of $f(z)$ of multiplicity $n$ if $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$. If $n = 1$, $z_0$ is called a simple pole.

A singular point $z_0$ of the function $f(z)$ is called an essential singularity of $f(z)$ if there exists no finite value of $n$ for which $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$.

**Theorem 1.** *The function $f$ has a pole of order $m$ at $z_0$ if and only if in some neighbourhood of $z_0$, $f$ can be expressed as*

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

*where $\phi(z)$ is analytic at $z_0$ and $\phi(z_0) \neq 0$.*

*Proof.* First assume that $z_0$ is a pole of $f$ of order $m$. Then in some neighbourhood of $z_0$, $f$ has a Laurent series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} b_n (z-z_0)^{-n}, \text{ where } b_m \neq 0.$$

Putting $\nu(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ we see that

$$\begin{aligned} f(z) &= \nu(z) + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ \ldots \ + \frac{b_m}{(z-z_0)^m} \\ &= \frac{(z-z_0)^m \nu(z) + b_1 (z-z_0)^{m-1} + \ \ldots \ + b_m}{(z-z_0)^m} \\ &= \frac{\phi(z)}{(z-z_0)^m}, \end{aligned}$$

where $\phi(z) = (z-z_0)^m \nu(z) + b_1(z-z_0)^{m-1} + \ \ldots \ + b_m$ is analytic at $z_0$ and $\phi(z_0) = b_m \neq 0$.

Next we assume that in some neighbourhood of $z_0$,

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where $\phi(z)$ is analytic at $z_0$ and $\phi(z_0) \neq 0$. Expanding $\phi(z)$ in Taylor series about $z_0$, we obtain

$$\begin{aligned} \phi(z) &= \sum_{n=0}^{\infty} a_n (z-z_0)^n \\ &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ \ldots \ + a_{m-1}(z-z_0)^{m-1} + \sum_{n=m}^{\infty} a_n (z-z_0)^n, \end{aligned}$$

where $a_0 = \phi(z_0) \neq 0$. Thus

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} = \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \ \ldots \ + \frac{a_{m-1}}{z-z_0} + \sum_{n=m}^{\infty} a_n (z-z_0)^{n-m},$$

which is the Laurent expansion of $f$ about $z_0$. Since $a_0 \neq 0$, it follows that $z_0$ is a pole of $f$ of order $m$. This completes the proof. □

**Theorem 2. (Riemann's Theorem)**

*If a function $f$ is bounded and analytic throughout a domain $0 <| z - z_0 |< \delta$, then $f$ is either analytic at $z_0$ or else $z_0$ is a removable singularity of $f$.*

*Proof.* Since $f$ is analytic throughout the domain $0 <| z - z_0 |< \delta$, $f$ can be represented in the Laurent series about $z_0$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}.$$

Let C denote the circle $| z - z_0 | = r \ (< \delta)$. Then putting $z - z_0 = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, we obtain

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz = \frac{r^n}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{in\theta} d\theta, \ n = 1, 2, \ \ldots$$

Since $f$ is bounded there exists a positive number $M$ such that $| f(z) |\leq M$ for all $z$ in the given domain. Therefore,

$$| b_n | = \frac{r^n}{2\pi} \left| \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{in\theta} d\theta \right| \leq \frac{r^n}{2\pi} \cdot 2\pi M = Mr^n \ for \ n = 1, 2, \ \ldots$$

Since $r$ can be chosen arbitrarily small, we have $b_n = 0$ for $n = 1, 2, \ \ldots$ Thus we obtain

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \ in \ 0 <| z - z_0 |< \delta.$$

This shows that $f$ is either analytic at $z_0$ or else $z_0$ is a removable singularity of $f$. This proves the theorem. □

**Theorem 3.** *If $z_0$ is a pole of the function $f$, then $\lim_{z \to z_0} f(z) = \infty$.*

*Proof.* Let $z_0$ be a pole of $f$ of order $m$. Then in some neighbourhood of $z_0$, we can write

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic at $z_0$ and $\phi(z_0) \neq 0$. $\phi(z)$ being analytic at $z_0$, it is continuous at $z_0$. Hence, for $\varepsilon = \frac{1}{2} | \phi(z_0) |> 0$, there exists a $\delta > 0$ such that

$$| \phi(z) - \phi(z_0) | < \varepsilon = \frac{1}{2} | \phi(z_0) | \ \ for \ \ | z - z_0 |< \delta.$$

Therefore,

$$\begin{aligned} | \phi(z) | &= | \phi(z) - \phi(z_0) + \phi(z_0) | \geq | \phi(z_0) | - | \phi(z) - \phi(z_0) | \\ &> | \phi(z_0) | - \frac{1}{2} | \phi(z_0) | = \frac{1}{2} | \phi(z_0) | \ \ for \ \ | z - z_0 |< \delta. \end{aligned}$$

Thus, for $\mid z - z_0 \mid < \delta$, we obtain $\mid f(z) \mid > \frac{\frac{1}{2}|\phi(z_0)|}{|z-z_0|^m}$. Let $G$ be a positive number, however large. Then $\mid f(z) \mid > G$

$$if \ \frac{\frac{1}{2} \mid \phi(z_0) \mid}{\mid z - z_0 \mid^m} > G \ and \ \mid z - z_0 \mid < \delta,$$

$$i.e. \ if \ \mid z - z_0 \mid < \left( \frac{\mid \phi(z_0) \mid}{2G} \right)^{1/m} \ and \ \mid z - z_0 \mid < \delta,$$

$$i.e. \ if \mid z - z_0 \mid < \delta_1 \ where \ \delta_1 = \min\{ \left( \frac{\mid \phi(z_0) \mid}{2G} \right)^{1/m}, \ \delta \}.$$

This means that $\lim_{z \to z_0} f(z) = \infty$. This proves the theorem. □

**Theorem 4.** *If $f(z)$ has an isolated singularity at $z = z_0$ and $f(z) \to \infty$ as $z \to z_0$, then $f(z)$ has a pole at $z = z_0$.*

*Proof.* Since $f(z) \to \infty$ as $z \to z_0$, for a given $R > 0$ there exists a $\delta > 0$ such that $f(z)$ is analytic for $0 < \mid z - z_0 \mid < \delta$ and

$$\mid f(z) \mid > R \ \ whenever \ \ 0 < \mid z - z_0 \mid < \delta.$$

In particular, $f(z) \neq 0$ for $0 < \mid z - z_0 \mid < \delta$ and so, $g(z) = 1/f(z)$ is analytic and bounded by $1/R$ in this deleted neighbourhood of $z_0$. Therefore by Riemann's theorem, $g(z)$ has a removable singularity at $z_0$, and we may write

$$g(z) = \frac{1}{f(z)} = a_1(z - z_0) + a_2(z - z_0)^2 + \ \ldots, \ 0 < \mid z - z_0 \mid < \delta.$$

Since $g(z) \neq 0$ for $0 < \mid z - z_0 \mid < \delta$, not all the coefficients of $g(z)$ are zero. This means that there is a $k \geq 1$ such that $a_k$ is the first nonzero coefficient of $g(z)$. Then

$$g(z) = \frac{1}{f(z)} = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \ \ldots,$$

so that

$$\begin{aligned} \frac{1}{(z - z_0)^k f(z)} &= a_k + a_{k+1}(z - z_0) + \ \ldots \\ &\to a_k \ \ as \ \ z \to z_0, \end{aligned}$$

and therefore,

$$\lim_{z \to z_0} (z - z_0)^k f(z) = \frac{1}{a_k} \neq 0.$$

This shows that $f(z)$ has a pole of order $k$ at $z = z_0$. This proves the theorem. □

**Example 1.** *Discuss singularities of the function*

$$f(z) = \frac{z}{z^2+4}.$$

**Solution.** *We have* $z^2+4 = (z+2i)(z-2i)$. *Therefore,* $f(z)$ *has singularities at* $z = 2i$ *and* $z = -2i$. *Since*

$$\lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{z(z-2i)}{(z+2i)(z-2i)} = \frac{1}{2} \neq 0,$$

$f(z)$ *has a simple pole at* $z = 2i$. *Again since,*

$$\lim_{z\to -2i}(z+2i)f(z) = \lim_{z\to -2i}\frac{z(z+2i)}{(z+2i)(z-2i)} = \frac{1}{2} \neq 0,$$

*it follows that,* $f(z)$ *has a simple pole at* $z = -2i$.

**Example 2.** *Classify the nature of singularity of the function*

$$f(z) = \frac{e^{-z}}{(z-3)^4}.$$

**Solution.** *We note that* $z = 3$ *is the only singularity of* $f(z)$. *To find the nature of singularity of* $f(z)$ *at* $z = 3$, *we expand* $f(z)$ *in a Laurent series valid in a deleted neighbourhood* $0 <\mid z-3 \mid< r$ *where* $r$ *is some positive number. Since*

$$\begin{aligned} f(z) &= \frac{e^{-z}}{(z-3)^4} = \frac{e^{-3}e^{-(z-3)}}{(z-3)^4} \\ &= e^{-3}\left[\frac{1}{(z-3)^4} - \frac{1}{(z-3)^3} + \frac{1}{2!(z-3)^2} - \frac{1}{3!(z-3)} + \ \dots\right], \end{aligned}$$

$f(z)$ *has a pole of order* $4$ *at* $z = 3$.

*Alternatively, the result follows from the fact that*

$$\lim_{z\to 3}(z-3)^4 f(z) = \lim_{z\to 3} e^{-z} = \frac{1}{e^3} \neq 0.$$